

# Weak Field Expansion of Gravity and Graphical Representation \*

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## Abstract

We introduce a graphical representation for a global SO(n) tensor  $\partial_\mu \partial_\nu h_{\alpha\beta}$ , which generally appears in the perturbative approach of gravity around the flat space:  $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ . We systematically construct global SO(n) invariants. Independence and completeness of those invariants are shown by taking examples of  $\partial\partial h$ - and  $(\partial\partial h)^2$ - invariants. They are classified graphically. Indices which characterize all independent invariants (or graphs) are given. We apply the results to general invariants with dimension  $(Mass)^4$  and the Gauss-Bonnet identity in 4-dim gravity.

## 1 Introduction

In n-dimensional Euclidean (Minkowskian) flat space(-time), fields are classified as scalar, spinor, vector, tensor, ... , by the transformation property under the global SO(n) ( SO(n-1,1) ) transformation of space(-time) coordinates.

$$x^{\mu'} = M^\mu_\nu x^\nu \quad , \quad (1)$$

where  $M$  is a  $n \times n$  matrix of  $SO(n)(SO(n-1,1))$ <sup>1</sup>. As for the lower spin fields, the field theory is well defined classically and quantumly.

The general curved space is described by the general relativity which is based on invariance under the general coordinate transformation. Its infinitesimal form is written as

$$\begin{aligned} x^{\mu'} &= x^\mu - \epsilon^\mu(x) \quad , \quad |\epsilon| \ll 1 \quad , \\ \delta g_{\mu\nu} &= g_{\mu\lambda} \nabla_\nu \epsilon^\lambda + g_{\nu\lambda} \nabla_\mu \epsilon^\lambda + O(\epsilon^2) = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \epsilon^\lambda + g_{\nu\lambda} \partial_\mu \epsilon^\lambda + O(\epsilon^2) \quad , \end{aligned} \quad (2)$$

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<sup>1</sup> Hereafter we take the Euclidean case for simplicity.

where  $\epsilon^\mu$  is an infinitesimally-small local free parameter. The general invariant composed of purely geometrical quantities and with the mass dimension (*Mass*)<sup>2</sup> is uniquely given by Riemann scalar curvature,  $R$ , defined by

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \quad , R_{\mu\nu\sigma}^\lambda = \partial_\nu \Gamma_{\mu\sigma}^\lambda + \Gamma_{\tau\nu}^\lambda \Gamma_{\mu\sigma}^\tau - (\nu \leftrightarrow \sigma) \quad , \\ R_{\mu\nu} = R_{\mu\nu\lambda}^\lambda \quad , \quad R = g^{\mu\nu}R_{\mu\nu} \quad , \quad g = +\det g_{\mu\nu} \quad . \quad (3)$$

It is well-known that the general relativity can be constructed purely within the flat space first by introducing a symmetric second rank tensor (Fierz-Pauli field) and then by requiring consistency in the field equation in a perturbative way of the weak field [1]. In the present case, we can obtain the perturbed lagrangian simply by the perturbation around the flat space.

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \quad . \quad (4)$$

Then the transformation (2) is expressed as

$$\delta h_{\mu\nu} = \partial_\mu \epsilon^\nu + h_{\mu\lambda} \partial_\nu \epsilon^\lambda + \frac{1}{2} \epsilon^\lambda \partial_\lambda h_{\mu\nu} + \mu \leftrightarrow \nu + O(\epsilon^2) \quad . \quad (5)$$

In the right-hand side (RHS), there appear  $h^0$ -order terms and  $h^1$ -order terms. Therefore the general coordinate transformation (5) does not preserve the weak-field ( $h_{\mu\nu}$ ) perturbation order. Riemann scalar curvature is also expanded as

$$R = \partial^2 h - \partial_\mu \partial_\nu h_{\mu\nu} - h_{\mu\nu}(\partial^2 h_{\mu\nu} - 2\partial_\lambda \partial_\mu h_{\nu\lambda} + \partial_\mu \partial_\nu h) \\ + \frac{1}{2}\partial_\mu h_{\nu\lambda} \cdot \partial_\nu h_{\mu\lambda} - \frac{3}{4}\partial_\mu h_{\nu\lambda} \cdot \partial_\mu h_{\nu\lambda} + \partial_\mu h_{\mu\lambda} \cdot \partial_\nu h_{\nu\lambda} \\ - \partial_\mu h_{\mu\nu} \cdot \partial_\nu h + \frac{1}{4}\partial_\mu h \cdot \partial_\mu h + O(h^3) \quad , \\ h \equiv h_{\mu\mu} \quad . \quad (6)$$

RHS is expanded into the infinite power series of  $h_{\mu\nu}$  due to the presence of the 'inverse' field of  $g_{\mu\nu}$ ,  $g^{\mu\nu}$ , in (3).

It is explicitly checked that  $R$ , defined perturbatively by the RHS of (6), transforms, under (5), as a scalar  $\delta R(x) = \epsilon^\lambda(x)\partial_\lambda R(x)$ , at the order of  $O(h)$ . Because the general coordinate symmetry does not preserve the the weak-field ( $h_{\mu\nu}$ ) perturbation order, we need  $O(h^2)$  terms in (6) in order to verify  $\delta R(x) = \epsilon^\lambda(x)\partial_\lambda R(x)$ , at the order of  $O(h)$ . The first two terms of RHS of (6),  $\partial^2 h$  and  $\partial_\mu \partial_\nu h_{\mu\nu}$ , are two independent global SO(n) invariants at the order  $O(h)$ . We may regard the weak field perturbation using (4) as a sort of 'linear' representation of the general coordinate symmetry, where all general invariant quantities are generally expressed by the infinite series of power of  $h_{\mu\nu}$ , and there appears no 'inverse' fields. One advantage of the linear representation is that the independence of invariants, as a local function of  $x^\mu$ , can be clearly shown because all quantities are written only by  $h_{\mu\nu}$  and its derivatives. We analyze some basic points of the weak-field expansion and develop a useful graphical technique.

Mathematically we classify all independent SO(n)-invariants of certain types, by use of the graph topology.

## 2 Representation of $\partial\partial h$ -tensors and invariants

We represent the 4-th rank global  $SO(n)$  tensor (4-tensor),  $\partial_\mu\partial_\nu h_{\alpha\beta}$ , as follows.

Fig.1 4-tensor  $\partial_\mu\partial_\nu h_{\alpha\beta}$

**Def 1** We call dotted lines *suffix-lines*, a rigid line a *bond*, a vertex with a crossing mark a *h-vertex* and that without it a *dd-vertex*.

This graph respects all suffix-permutation symmetries of  $\partial_\mu\partial_\nu h_{\alpha\beta}$  :

$$\partial_\mu\partial_\nu h_{\alpha\beta} = \partial_\nu\partial_\mu h_{\alpha\beta} = \partial_\mu\partial_\nu h_{\beta\alpha} . \quad (7)$$

**Def 2** The suffix *contraction* is expressed by connecting the two corresponding suffix-lines.

For example, 2-tensors :  $\partial^2 h_{\alpha\beta}$  ,  $\partial_\mu\partial_\nu h_{\alpha\alpha}$  ,  $\partial_\mu\partial_\beta h_{\alpha\beta}$  , which are made from Fig.1 by connecting two suffix-lines, are expressed as in Fig.2.

Fig.2 2-tensors of (a)  $\partial^2 h_{\alpha\beta}$  , (b)  $\partial_\mu\partial_\nu h_{\alpha\alpha}$  and (c)  $\partial_\mu\partial_\beta h_{\alpha\beta}$

Two independent invariants (0-tensors) :  $P \equiv \partial_\mu\partial_\mu h_{\alpha\alpha}$ ,  $Q \equiv \partial_\alpha\partial_\beta h_{\alpha\beta}$  , which are made from Fig.2 by connecting the remaining two suffix-lines, are expressed as in Fig.3.

Fig.3 Invariants of  $P \equiv \partial_\mu\partial_\mu h_{\alpha\alpha}$  and  $Q \equiv \partial_\alpha\partial_\beta h_{\alpha\beta}$  .

$P$  and  $Q$  are all possible invariants of  $\partial\partial h$ -type. All suffix-lines of Fig.3 are closed. We easily see the following lemma is valid.

**Lemma 1** Generally all suffix-lines of invariants are *closed*. We call a closed suffix-line a *suffix-loop*.

## 3 Representation of $(\partial\partial h)^2$ -tensors and invariants

Now we begin to deal with 'products' of two  $\partial\partial h$ -tensors. As examples of  $SO(n)$ -tensors, we have the representations of Fig.4 for  $\partial_\mu\partial_\nu h_{\alpha\beta}\partial_\mu\partial_\nu h_{\gamma\delta}$  and  $\partial_\mu\partial_\nu h_{\alpha\beta}\partial_\nu\partial_\lambda h_{\lambda\beta}$ .

Fig.4 Graphical Representations of  $\partial_\mu\partial_\nu h_{\alpha\beta} \cdot \partial_\mu\partial_\nu h_{\gamma\delta}$  and  $\partial_\mu\partial_\nu h_{\alpha\beta} \cdot \partial_\nu\partial_\lambda h_{\lambda\beta}$ .

Before listing up all possible  $(\partial\partial h)^2$ -invariants, let us state a lemma on a general  $SO(n)$ -invariant made of  $s$   $\partial\partial h$ -tensors.

**Lemma 2** Let a general  $(\partial\partial h)^s$ -invariant ( $s = 1, 2, \dots$ ) has  $l$  suffix-loops. Let each loop have  $v_i$  h-vertices and  $w_i$  dd-vertices ( $i = 1, 2, \dots, l - 1, l$ ). We have the following *necessary* conditions for  $s, l, v_i$  and  $w_i$ .

$$\begin{aligned} \sum_{i=1}^l v_i &= s \quad , \quad \sum_{i=1}^l w_i = s \quad , \\ v_i \geq 0 \quad , \quad w_i \geq 0 \quad , \quad v_i + w_i \geq 1 \quad , \\ v_i, w_i &= 0, 1, 2, \dots \quad , \quad l = 1, 2, 3, \dots, 2s - 1, 2s \quad . \end{aligned} \quad (8)$$

Here we may ignore the ordering of the elements in a set  $\left\{ \begin{pmatrix} v_i \\ w_i \end{pmatrix} ; i = 1, 2, \dots, l - 1, l \right\}$  because the order can be arbitrarily changed by renumbering the suffix-loops.

This lemma is valid because the considered graph is made by contracting all sufix-lines of  $s$  4-tensors of Fig.1. We use the above Lemma for the case  $s = 2$  to list up all possible  $(\partial\partial h)^2$ -invariants.

(i)  $l = 1$

For this case, we have

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (9)$$

There are two ways to distribute two dd-vertices and two h-vertices on one suffix-loop. See Fig.5, where a small circle is used to represent a dd-vertex explicitly.

Fig.5 Two ways to distribute two dd-vertices (small circles) and two h-vertices (cross marks) upon one suffix-loop.

**Def 3** We call diagrams without bonds, like Fig.5, *bondless diagrams*.

Finally, taking account of the two bonds, we have three independent  $(\partial\partial h)^2$ -invariants for the case  $l = 1$ . We name them  $A1, A2$  and  $A3$  as shown in Fig.6.

Fig.6 Three independent  $(\partial\partial h)^2$ -invariants for the case of one suffix-loop.

(ii)  $l = 2$

For this case, we have

$$\begin{aligned} \left\{ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \right\} &= (a) : \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad , \quad (b) : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \\ (c) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad , \quad (d) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \end{aligned} \quad (10)$$

where the ordering of  $\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$  and  $\begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$  is irrelevant for the present classification as stated in Lemma 2.<sup>2</sup> Each one above has one bondless diagram as shown in Fig.7.

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<sup>2</sup> The same treatment is adopted in the following other cases.

Fig.7 Bondless diagrams for (10).

Then we have 5 independent  $(\partial\partial h)^2$ -invariants for this case  $l = 2$ . We name them  $B1, B2, B3, B4$  and  $QQ$  as shown in Fig.8. Among them  $QQ$  is a *disconnected diagram*. Fig.7b has two independent ways to connect vertices by two bonds.

Fig.8 Five independent  $(\partial\partial h)^2$ -invariants for the case of two suffix-loops.

(iii)  $l = 3$

For this case, we have

$$\begin{aligned} \left\{ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \right\} &= (a) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} , \\ (b) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} , \quad (c) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \end{aligned} \quad (11)$$

Each one above has one bondless diagram as shown in Fig.9.

Fig.9 Three bondless diagrams corresponding to (11).

Then we have 4 independent  $(\partial\partial h)^2$ -invariants for the case  $l = 3$ . We name them  $C1, C2, C3$ , and  $PQ$  as shown in Fig.10. Among them  $PQ$  is a disconnected diagram. Fig.9c has two independent ways to connect vertices by two bonds.

Fig.10 Four independent  $(\partial\partial h)^2$ -invariants for the case of three suffix-loops.

(iv)  $l = 4$

For this case, we have

$$\left\{ \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \begin{pmatrix} v_4 \\ w_4 \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \quad (12)$$

This corresponds to one bondless diagram shown in Fig.11.

Fig.11 The bondless diagram corresponding to (12).

Then we have a unique independent  $(\partial\partial h)^2$ -invariant (disconnected) for the case  $l = 4$ . We name it  $PP$  as shown in Fig.12.

Fig.12 The unique independent  $(\partial\partial h)^2$ -invariant for the case of four suffix-loops.

We have obtained  $3(l = 1) + 5(l = 2) + 4(l = 3) + 1(l = 4) = 13$   $(\partial\partial h)^2$ -invariants from the necessary conditions (8), Lemma 2. ( Among them 3 ones ( $QQ, PQ, PP$ ) are disconnected.) Their independence is assured by their difference of the connectivity of suffix-lines, in other words, the topology of the graphs. Therefore, to conclude this section, we have completely listed up all independent  $(\partial\partial h)^2$ -invariants. The ordinary mathematical expressions for the 13 invariants will be listed in Table 1 of Sec.5. In the next section, we reprove the completeness of the above enumeration from the standpoint of a suffix-permutation symmetry and the combinatorics among suffixes.

## 4 Completeness of Graph Enumeration

Let us examine the SO(n)-invariants listed in Sec.2 and Sec.3 from the viewpoint of the suffix-permutation symmetry (7).

### (i) $\partial\partial h$ -invariants

The  $\partial\partial h$ -invariants are obtained by contracting 4 indices  $(\mu_1, \mu_2, \mu_3, \mu_4)$  in  $\partial_{\mu_1}\partial_{\mu_2}h_{\mu_3\mu_4}$ . All possible ways of contracting the four indices are given by the following 3 ones.

$$a) \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}, \quad b) \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}, \quad c) \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}. \quad (13)$$

Due to the symmetry (7), we see b) and c) give the same invariant Q.

**Def 4** We generally call the number of occurrence of a covariant (which includes the case of an invariant) C, when contracting suffixes of a covariant  $C'$  in all possible ways, a *weight* of C from  $C'$ .

In the present case, P has a weight 1 and Q has a weight 2 (from 4-tensor  $\partial_\mu\partial_\nu h_{\alpha\beta}$ ). We have an identity between the number of all possible ways of suffix-contraction (13) and weights of invariants.

$$3 = 1(P) + 2(Q). \quad (14)$$

A weight of an invariant shows 'degeneracy' in the contraction due to its suffix-permutation symmetry. The above identity shows the completeness of the enumeration of  $\partial\partial h$ -invariants from the viewpoint of the permutation symmetry.

### (ii) $(\partial\partial h)^2$ -invariants

We can do the same analysis for  $(\partial\partial h)^2$ -invariants. The number of all possible contraction of 8 indices in the 8-tensor  $\partial_{\mu_1}\partial_{\mu_2}h_{\mu_3\mu_4} \cdot \partial_{\mu_5}\partial_{\mu_6}h_{\mu_7\mu_8}$  is  $7 \times 5 \times 3 \times 1 = 105$ . Let us take B1 of Fig.8c as an example of weight calculation. See Fig. 13.

$$\begin{aligned} \text{The weight of B1 from the 8-tensor} &= 1(\text{weight of Fig.2b from 4-tensor } \partial\partial h) \\ &\times 4(\text{weight of Fig.2c from 4-tensor } \partial\partial h) \times 2(\text{two ways of 2b-2c contraction}) \\ &\times 2(\text{two ways of choosing 2b-bond and 2c-bond among 2 bonds}) = 16. \end{aligned} \quad (15)$$

Fig.13 Graph B1 for the weight calculation (15).

Similarly we can obtain weights for all other  $(\partial\partial h)^2$ -invariants and the following identity holds true.

$$\begin{aligned} 105 &= 7 \times 5 \times 3 \times 1 = 16(A1) + 16(A2) + 16(A3) \\ &+ 16(B1) + 16(B2) + 4(B3) + 4(B4) + 4(QQ) \\ &+ 2(C1) + 2(C2) + 4(C3) + 4(PQ) + 1(PP). \end{aligned} \quad (16)$$

This identity clearly shows the completeness of the 13  $(\partial\partial h)^2$ -invariants listed in Sect.3.

Weights, defined above, correspond to the symmetry factor or the statistical factor in the Feynman diagram expansion of the field theory. Further the above identity (16) reminds us of a similar one, in the graph theory, called 'Polya's enumeration theorem'[5].

## 5 Indices for Graphs

The graph representation is very useful in proving mathematical properties, such as completeness and independence, of  $\text{SO}(n)$  invariants because the connectivity of suffixes can be read in the topology of a graph. In practical calculation, however, depicting graphs is cumbersome. In order to specify every graph of invariant succinctly, we present a set of indices which shows how suffix-lines (suffixes) within one  $\partial\partial h$  or two  $\partial\partial h$ 's are connected (contracted). In this section we characterize every independent graph of invariant by a set of some indices.<sup>3</sup>

(i) Number of Suffix Loops ( $\underline{l}$ )<sup>4</sup>

The number of suffix loops ( $\underline{l}$ ) of a graph is a good index. In fact, every  $\partial\partial h$ -invariant is completely characterized by  $\underline{l}$ :  $\underline{l}=2$  for P and  $\underline{l}=1$  for Q. The index  $\underline{l}$  is not sufficient to discriminate every  $(\partial\partial h)^2$ -invariant. We need the following ones, (ii) and (iii).

(ii) Number of Tadpoles (tadpoleno) and Type of Tadpole (tadtype[ ])

**Def 5** We call a closed suffix-loop which has only one vertex, a *tadpole*. When the vertex is dd-vertex (h-vertex), its *tadpole type*, tadtype[ ], is defined to be 0 (1). tadtype[ ] is assigned for each tadpole. The number of tadpoles which a graph has, is called *tadpole number* (tadpoleno) of the graph.

For example, in Fig.3, P has tadpoleno=2 and tadtype[1]=0 and tadtype[2]=1<sup>5</sup>. Q has tadpoleno=0.

(iii) Bond Changing Number(bcn[ ]) and Vertex Changing Number(vcn[ ])

**Def 6** bcn[ ] and vcn[ ] are defined for each suffix-loop as follows. When we trace a suffix-loop, starting from a vertex in a certain direction, we generally pass some vertices, and finally come back to the starting vertex. See Fig.14. When we move, in the tracing, from a vertex to a next vertex, we compare the bonds to which the two vertices belong, and their vertex-types. If the bonds are different, we set  $\Delta_{bcn} = 1$ , otherwise  $\Delta_{bcn} = 0$ . If the vertex-types are different, we set  $\Delta_{vcn} = 1$ , otherwise  $\Delta_{vcn} = 0$ . For  $l$ -th loop, we sum every number of  $\Delta_{bcn}$  and  $\Delta_{vcn}$  while tracing the loop once, and assign as  $\sum_{\text{along } l\text{-th loop}} \Delta_{bcn} \equiv \underline{bcn}[l]$ ,  $\sum_{\text{along } l\text{-th loop}} \Delta_{vcn} \equiv \underline{vcn}[l]$ <sup>6</sup>.

Fig.14 Explanation of bcn[ ] and vcn[ ] using Graph A2.

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<sup>3</sup> This approach is very contrasting with a standard one in the graph theory where the incidence matrix or the adjacency matrix are used for specifying a graph[5, 6].

<sup>4</sup> All indices are underlined. For example,  $\underline{l}$ , tadpoleno, tadtype, bcn, and vcn.

<sup>5</sup> Final results should be independent of arbitrariness in numbering all tadpoles in a graph.

<sup>6</sup> Final results should be independent of arbitrariness of numbering all suffix-loops in a graph.

Practically we calculate bcn[ ] and vcn[ ] as explained in Appendix A.

In Table 1, we list all indices necessary for discriminating every  $(\partial\partial h)^2$ -invariant completely.

Graph \ Indices	<u><math>l</math></u>	<u>tadpoleno</u>	<u>tadtype[ ]</u>	<u>bcn[ ]</u>	<u>vcn[ ]</u>
$A1 = \partial_\sigma \partial_\lambda h_{\mu\nu} \cdot \partial_\sigma \partial_\nu h_{\mu\lambda}$	1	0	nothing	4	2
$A2 = \partial_\sigma \partial_\lambda h_{\lambda\mu} \cdot \partial_\sigma \partial_\nu h_{\mu\nu}$	1	0	nothing	2	2
$A3 = \partial_\sigma \partial_\lambda h_{\lambda\mu} \cdot \partial_\mu \partial_\nu h_{\nu\sigma}$	1	0	nothing	2	4
$B1 = \partial_\nu \partial_\lambda h_{\sigma\sigma} \cdot \partial_\lambda \partial_\mu h_{\mu\nu}$	2	1	1	/	/
$B2 = \partial^2 h_{\lambda\nu} \cdot \partial_\lambda \partial_\mu h_{\mu\nu}$	2	1	0	/	/
$B3 = \partial_\mu \partial_\nu h_{\lambda\sigma} \cdot \partial_\mu \partial_\nu h_{\lambda\sigma}$	2	0	nothing	2	0
				2	0
$B4 = \partial_\mu \partial_\nu h_{\lambda\sigma} \cdot \partial_\lambda \partial_\sigma h_{\mu\nu}$	2	0	nothing	2	2
				2	2
$Q^2 = (\partial_\mu \partial_\nu h_{\mu\nu})^2$	2	0	nothing	0	2
				0	2
$C1 = \partial_\mu \partial_\nu h_{\lambda\lambda} \cdot \partial_\mu \partial_\nu h_{\sigma\sigma}$	3	2	1	/	/
$C2 = \partial^2 h_{\mu\nu} \cdot \partial^2 h_{\mu\nu}$	3	2	0	/	/
$C3 = \partial_\mu \partial_\nu h_{\lambda\lambda} \cdot \partial^2 h_{\mu\nu}$	3	2	1	0	0
				0	0
				2	2
$PQ = \partial^2 h_{\lambda\lambda} \cdot \partial_\mu \partial_\nu h_{\mu\nu}$	3	2	0	1	0
				0	0
				0	2
$P^2 = (\partial^2 h_{\lambda\lambda})^2$	4	/	/	/	/

Table 1 List of indices for all  $(\partial\partial h)^2$ -invariants. The symbol '/' means 'need not be calculated for discrimination'.

The listed 13 invariants are independent each other because Table 1 clearly shows

the topology of every graph is different.

## 6 Application to Gravitational Theories

Let us apply the obtained result to some simple problems. First Riemann tensors are graphically represented as in Fig.15.

Fig.15 Graphical representation of weak expansion of Rieman tensors .

Using them, general invariants with the mass dimension  $(\text{Mass})^4$  are expanded as in Table 2.

Graph	$\nabla^2 R$	$R^2$	$R_{\mu\nu}R^{\mu\nu}$	$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$
$A1$	1	0	0	-2
$A2$	2	0	$\frac{1}{2}$	0
$A3$	0	0	$\frac{1}{2}$	0
$B1$	-2	0	-1	0
$B2$	2	0	-1	0
$B3$	$-\frac{3}{2}$	0	0	1
$B4$	0	0	0	1
$Q^2$	0	1	0	0
$C1$	$\frac{1}{2}$	0	$\frac{1}{4}$	0
$C2$	-1	0	$\frac{1}{4}$	0
$C3$	-1	0	$\frac{1}{2}$	0
$PQ$	0	-2	0	0
$P^2$	0	1	0	0

Table 2 Weak-Expansion of Invariants with (Mass)<sup>4</sup>-Dim. :  $(\partial\partial h)^2$ -Part

The four invariants,  $\nabla^2 R$ ,  $R^2$ ,  $R_{\mu\nu}R^{\mu\nu}$  and  $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ , are important in the Weyl anomaly calculation[7, 3] and (1-loop) counter term calculation in 4 dim quantum gravity[8, 9]. From the explicit result of Table 2, we see the four invariants are independent as local functions of  $h_{\mu\nu}(x)$ , because the 13  $(\partial\partial h)^2$ -invariants are independent each other. In particular, the three 'products' of Riemann tensors ( $R^2$ ,  $R_{\mu\nu}R^{\mu\nu}$ ,  $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ ) are 'orthogonal', at the leading order of weak field perturbation, in the space 'spanned' by the 13  $(\partial\partial h)^2$ -invariants. Note here that the independence of the four invariants is proven for a general metric  $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ .

The four general invariants above are independent and complete as the Weyl anomaly terms. In the counter term calculation, however, we must take into account the arbitrariness of total derivative terms, because the counter term  $\Delta\mathcal{L}$  is usually defined in the action as

$$\int d^4x \Delta\mathcal{L} , \quad (17)$$

and fields  $h_{\mu\nu}(x)$  are usually assumed to damp sufficiently rapidly at a boundary. A manifest total derivative term is  $\sqrt{g}\nabla^2 R$ .

$$\int d^4x \sqrt{g}\nabla^2 R = \int d^4x \frac{\partial}{\partial x^\mu}(\sqrt{g}\nabla^\mu R) . \quad (18)$$

A nontrivial one is the Gauss-Bonnet topological quantity:  $R_{\mu\nu\alpha\beta}R_{\lambda\sigma\gamma\delta}\epsilon^{\mu\nu\lambda\sigma}\epsilon^{\alpha\beta\gamma\delta}/4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \equiv I_{GB}^{4d}$  where  $\epsilon^{\mu\nu\lambda\sigma}$  is the totally antisymmetric constant tensor ( $\epsilon^{1234} = 1$ ). From Table 2, we obtain

$$\begin{aligned} \sqrt{g}I_{GB}^{4d} &= I_{GB}^{4d} + O'(h^3) \\ &= -2(A1) - 2(A2) - 2(A3) + 4(B1) + 4(B2) + (B3) + (B4) \\ &\quad + (Q^2) - (C1) - (C2) - 2(C3) - 2(PQ) + (P^2) + O(h^3) \end{aligned} \quad (19)$$

$$\begin{aligned}
&= \partial_\mu \partial_\alpha h_{\nu\beta} \cdot \partial_\lambda \partial_\gamma h_{\sigma\delta} \cdot \epsilon^{\mu\nu\lambda\sigma} \epsilon^{\alpha\beta\gamma\delta} + O(h^3) \\
&= \partial_\mu (\partial_\alpha h_{\nu\beta} \cdot \partial_\lambda \partial_\gamma h_{\sigma\delta} \cdot \epsilon^{\mu\nu\lambda\sigma} \epsilon^{\alpha\beta\gamma\delta}) + O(h^3) \quad .
\end{aligned}$$

Surely  $I_{GB}^{4d}$  can be expressed in a form of a total derivative term. Therefore we can take, as the independent 1-loop counter terms in 4-dim pure Einstein quantum gravity, the following two terms[8].

$$R^2 \quad , \quad R_{\mu\nu} R^{\mu\nu} \quad . \quad (20)$$

## 7 Conclusions and Discussions

We have presented a graphical representation of global SO(n) tensors. This approach allows us to systematically list up all and independent SO(n) invariants. The completeness of the list is reassured by an identity between a combinatoric number of suffixes and weights of listed terms due to their suffix-permutation symmetries. Some indices, sufficient for discriminating all  $\partial\partial h$ - and  $(\partial\partial h)^2$ -invariants, are given. They are useful in practical (computer) calculation. Finally we have applied the result to some simple problems in the general relativity.

The present graphical representation for global SO(n) tensors is complementary to that for general tensors given in [2]. The latter one deals with only general covariants, and its results are independent of the perturbation. In the covariant representation, however, it is difficult to prove the independence of listed general invariants because there is no independent 'bases'. On the other hand, in the present case, although the analysis is based on the weak field perturbation, we have independent 'bases' (like 13  $(\partial\partial h)^2$ -invariants) at each perturbation order. It allows us to prove independence of listed invariants.

Stimulated by the duality properties of superstring theories, anomaly structure of supergravities in higher dimensions (say, 6 dim and 10 dim) recently becomes important. Generally in n-dim gravity, Weyl anomaly is given by some combination of general invariants with dimension  $(\text{Mass})^n$  and  $L$ -loop counter-terms are given by some combination of general invariants with dimension  $(\text{Mass})^{n+2L-2}$ . The present approach will be useful in those explicit calculation. The case for 6 dim has been analyzed in [4].

Some results such as (16),(19) and Table 2 are obtained or checked by the computer calculation using a C-language program [10].

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## Appendix A. Calculation of $\underline{bcn}[\ ]$ and $\underline{vcn}[\ ]$

We explain how to calculate the indices,  $\underline{bcn}[\ ]$  and  $\underline{vcn}[\ ]$  in the actual(computer) calculation. Let us consider a  $(\partial\partial h)^2$ -invariant. It has two bonds. As an example, we take C1 in Fig.16.

**Def 7** We assign  $i=0$  for one bond and  $i=1$  for the other. 'i' is the *bond number* and discriminates the two bonds. Next we assign  $j=0$  for all dd-vertices and  $j=1$  for all h-vertices. 'j' is the *vertex-type number* and discriminate the vertex-type. Any vertex in a graph is specified by a pair  $(i,j)$ .

Fig.16 Bond number 'i' and vertex-type number 'j' for each vertex  
in the invariant C1.

**Def 8** When we trace a suffix-line, along a loop, starting from a vertex  $(i_0,j_0)$  in a certain direction, we pass some vertices,  $(i_1,j_1), (i_2,j_2), \dots$  and finally come back to the starting vertex  $(i_0,j_0)$ . We focus on the *change* of the bond number, i, and the vertex-type number, j, when we pass from a vertex to the next vertex in the tracing (see Fig.17 and 14). For  $l$ -th loop, we assign as  $\sum_{\text{along } l\text{-th loop}} |\Delta i| \equiv \underline{bcn}[l]$ ,  $\sum_{\text{along } l\text{-th loop}} |\Delta j| \equiv \underline{vcn}[l]$ .

Fig. 17 Change of i (bond number) and j (vertex-type number).  
Arrows indicate directions of tracings.

$\underline{bcn}[\ ]$  and  $\underline{vcn}[\ ]$  are listed for all  $(\partial\partial h)^2$ -invariants in Table 1.  $\underline{bcn}[\ ]$  and  $\underline{vcn}[\ ]$  defined above satisfy the following important properties.

1. They do not depend on the starting vertex for tracing along a loop.
2. They donot depend on the direction of the tracing.

## Appendix B. Gauge-Fixing Condition and Graphical Rule

In the text, we have not taken a gauge-fixing condition. When we calculate a physical quantity in the classical and quantum gravity, we sometimes need to impose the condition on the metric  $g_{\mu\nu}$  for some reasons. Firstly, in the case of quantizing gravity itself or of solving a classical field equation with respect to the gravity mode, we *must* impose the fixing condition in order to eliminate the local freedom ( $\epsilon^\mu(x)$ ,  $\mu = 1, 2, \dots, n-1, n$ ) due to the general coordinate invariance (2):  $g_{\mu\nu} \rightarrow g_{\mu\nu} + g_{\mu\lambda}\nabla_\nu\epsilon^\lambda + g_{\nu\lambda}\nabla_\mu\epsilon^\lambda$ . Secondly, even when the condition is theoretically not necessary (such as the quantization on the fixed curved space, or the ordinary anomaly calculation), the gauge-fixing is practically useful because it considerably reduces the number of  $SO(n)$  invariants to be considered.

In the weak gravity case  $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ ,  $|h_{\mu\nu}| \ll 1$ , the condition is expressed by  $h_{\mu\nu}$ . Let us take a familiar gauge:

$$\partial_\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h \quad , \quad h \equiv h_{\lambda\lambda} \quad . \quad (21)$$

This condition leads to the following condition on the present basic element  $\partial_\mu\partial_\nu h_{\alpha\beta}$ .

$$\partial_\lambda\partial_\mu h_{\mu\nu} = \frac{1}{2}\partial_\lambda\partial_\nu h \quad , \quad h \equiv h_{\lambda\lambda} \quad . \quad (22)$$

This gives us a graphical rule shown in Fig.18.

Fig.18 Graphical rule, expressing (22), due to the gauge-fixing condition (21) .

Let us see how does this rule reduce the number of independent invariants given in the text. For  $\partial\partial h$ -invariants, we obtain the following relation

$$Q = \frac{1}{2}P \quad . \quad (23)$$

For  $(\partial\partial h)^2$ -invariants, we obtain the following relations.

$$\begin{aligned} A2 &= A3 = \frac{1}{2}B1 = \frac{1}{4}C1 \quad , \\ B2 &= \frac{1}{2}C3 \quad , \quad QQ = \frac{1}{2}PQ = \frac{1}{4}PP \quad . \end{aligned} \quad (24)$$

Therefore, in the gauge (21), we can reduce the number of independent invariants from 2 to 1 for  $\partial\partial h$ -invariants (say,  $P$ ) and from 13 to 7 for  $(\partial\partial h)^2$ -invariants (say,  $A1, B3, B4, C1, C2, C3, PP$ ).

We expect this gauge-fixed treatment is practically very useful when a calculating quantity is guaranteed to be gauge-invariant in advance.

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## Figure Captions

- Fig.1 4-tensor  $\partial_\mu \partial_\nu h_{\alpha\beta}$
- Fig.2 2-tensors of  $\partial^2 h_{\alpha\beta}$ ,  $\partial_\mu \partial_\nu h_{\alpha\alpha}$  and  $\partial_\mu \partial_\beta h_{\alpha\beta}$
- Fig.3 Invariants of  $P \equiv \partial_\mu \partial_\mu h_{\alpha\alpha}$  and  $Q \equiv \partial_\alpha \partial_\beta h_{\alpha\beta}$ .
- Fig.4 Graphical Representations of  $\partial_\mu \partial_\nu h_{\alpha\beta} \partial_\mu \partial_\nu h_{\gamma\delta}$  and  $\partial_\mu \partial_\nu h_{\alpha\beta} \partial_\nu \partial_\lambda h_{\lambda\beta}$ .
- Fig.5 Two ways to distribute two dd-vertices ( small circles) and two h-vertices (cross marks) upon one suffix-loop.
- Fig.6 Three independent  $(\partial\partial h)^2$ -invariants for the case of one suffix-loop.
- Fig.7 Bondless diagrams for (10).
- Fig.8 Five independent  $(\partial\partial h)^2$ -invariants for the case of two suffix-loops.
- Fig.9 Three bondless diagrams corresponding to (11).
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- Fig.11 The bondless diagram corresponding to (12).
- Fig.12 One independent  $(\partial\partial h)^2$ -invariant for the case of four suffix-loops.
- Fig.13 Graph B1 for the weight calculation (15).
- Fig.14 Explanation of bcn[ ] and vcn[ ] using Graph A2.
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Arrows indicate directions of tracings.
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